CONNECTIONS ON MODULES OVER SINGULARITIES OF FINITE CM REPRESENTATION TYPE

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Abstract. Let A be a commutative k-algebra, where k is an algebraically closed field of characteristic 0, and let M be an A-module. We consider the following question: Under what conditions on A and M is it possible to find a connection $\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ on M?

We consider maximal Cohen-Macaulay (MCM) modules over complete CM algebras that are isolated singularities, and usually assume that the singularities have finite CM representation type. It is known that any MCM module over a simple singularity of dimension $d \leq 2$ admits an integrable connection. We prove that an MCM module over a simple singularity of dimension $d \geq 3$ admits a connection if and only if it is free. Among singularities of finite CM representation type, we find examples of curves with MCM modules that do not admit connections, and threefolds with non-free MCM modules that admit connections.

Let A be a singularity not necessarily of finite CM representation type, and consider the condition that A is a Gorenstein curve or a $\mathbf{Q}$-Gorenstein singularity of dimension $d \geq 2$. We show that this condition is sufficient for the canonical module ω_A to admit an integrable connection, and conjecture that it is also necessary. In support of the conjecture, we show that if A is a monomial curve singularity, then the canonical module ω_A admits an integrable connection if and only if A is Gorenstein.

Introduction

Let k be an algebraically closed field of characteristic 0, and let A be a commutative k-algebra. For any A-module M, we consider the notion of a *connection* on M, i.e. an A-linear homomorphism

$$\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$$

such that $\nabla_D(am) = a\nabla_D(m) + D(a)m$ for all $D \in \mathrm{Der}_k(A)$, $a \in A$ and $m \in M$. A connection is integrable if it is a Lie algebra homomorphism. The present paper is devoted to the following question: Under what conditions on A and M is it possible to find a connection on M?

We consider a complete local CM k-algebra A with residue field k that is an isolated singularity, and a maximal Cohen-Macaulay (MCM) A-module M. Moreover, we usually assume that A has finite CM representation type, i.e. the number of isomorphism classes of indecomposable MCM A-modules is finite.

If A is a hypersurface, then A has finite CM representation type if and only if it is a simple singularity. By convention, $A = k[[x]]/(x^{n+1})$ for $n \geq 1$ are the simple singularities (of type A_n) of dimension zero. It is known that if A is a simple

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singularity of dimension $d \leq 2$, then any MCM A-module admits an integrable connection, see section 5. We prove the following result:

Theorem 1. *Let A be the complete local ring of a simple singularity of dimension $d \geq 3$. Then an MCM A-module M admits a connection if and only if M is free.*

A Gorenstein singularity with finite CM representation type is a hypersurface, and there is a complete classification of non-Gorenstein singularities of finite CM representation type in dimension $d \leq 2$:

(1) The curve singularities D_n^s for $n \geq 2$ and E_6^s, E_7^s, E_8^s.
(2) The quotient surface singularities that are non-Gorenstein.

On the other hand, the classification is not complete in higher dimensions. The only known examples of non-Gorenstein singularities of finite CM representation type in dimension $d \geq 3$ are the following threefolds:

(3) The quotient threefold singularity of type $\frac{1}{2}(1,1,1)$.
(4) The threefold scroll of type $(2,1)$.

Definitions of the singularities in (1)-(4) are given in section 5.

Among the curve singularities of finite CM representation type, we find examples of singularities with MCM modules that do not admit connections. In fact, the canonical module ω_A does not admit a connection when A is the complete local ring of one of the singularities E_6^s, E_7^s, E_8^s. Moreover, it seems that the same holds for the singularities D_n^s for all $n \geq 2$.

However, for the surface singularities of finite CM representation type, all MCM modules admit connections. This is a consequence of the fact that these singularities are quotient singularities, and that all MCM modules are induced by group representations. It is perhaps more surprising that it is difficult to find examples of MCM modules over surface singularities that do not admit connections, even when the singularities have infinite CM representation type.

In dimension $d \geq 3$, we would like to find examples of non-free MCM modules that admit connections. We find that over the threefold scroll of type $(2,1)$, no non-free MCM modules admit connections. However, over the threefold quotient singularity of type $\frac{1}{2}(1,1,1)$, the canonical module admits an integrable connection.

Let A be a singularity not necessarily of finite CM representation type. Let us consider the condition that A is a Gorenstein curve or a $\mathbf{Q}$-Gorenstein singularity of dimension $d \geq 2$. We show that this condition is sufficient for the canonical module ω_A to admit an integrable connection. In accordance with all our results for singularities of finite CM representation type, we make the following conjecture:

Conjecture 2. *Let A be the complete local ring of a singularity of dimension $d \geq 1$. Then the canonical module ω_A admits a connection if and only if A is a Gorenstein curve or a $\mathbf{Q}$-Gorenstein singularity of dimension $d \geq 2$.*

Let us consider a monomial curve singularity A not necessarily of finite CM representation type. We show that if A is Gorenstein, then any gradable rank one MCM A-module admits an integrable connection. We also prove the following theorem, in support of our conjecture:

Theorem 3. *Let A be the complete local ring of a monomial curve singularity. Then the canonical A-module ω_A admits a connection if and only if A is Gorenstein.*

1. Basic definitions

Let k be an algebraically closed field of characteristic 0, and let A be a commutative k-algebra. A *Lie-Rinehart algebra* of A/k is a pair (g, τ), where g is an A-module and a k-Lie algebra, and $\tau : \mathsf{g} \to \mathrm{Der}_k(A)$ is a morphism of A-modules and k-Lie algebras, such that

$$[D, aD'] = a[D, D'] + \tau_D(a)\, D'$$

for all $D, D' \in \mathsf{g}$ and all $a \in A$, see Rinehart [27]. A Lie-Rinehart algebra is the algebraic analogue of a *Lie algebroid*, and it is also known as a Lie pseudo-algebra or a Lie-Cartan pair.

When g is a subset of $\mathrm{Der}_k(A)$ and $\tau : \mathsf{g} \to \mathrm{Der}_k(A)$ is the inclusion map, the pair (g, τ) is a Lie-Rinehart algebra if and only if g is closed under the A-module and k-Lie algebra structures of $\mathrm{Der}_k(A)$. We are mainly interested in Lie-Rinehart algebras of this type, and usually omit τ from the notation.

Let g be a Lie-Rinehart algebra. For any A-module M, we define a g*-connection* on M to be an A-linear map $\nabla : \mathsf{g} \to \mathrm{End}_k(M)$ such that

$$\nabla_D(am) = a\nabla_D(m) + D(a)\, m \tag{1}$$

for all $D \in \mathsf{g},\ a \in A,\ m \in M$. We say that ∇ satisfies the *derivation property* when condition (1) holds for all $D \in \mathsf{g}$. If $\nabla : \mathsf{g} \to \mathrm{End}_k(M)$ is a k-linear map that satisfies the derivation property, we call ∇ a k-linear g-connection on M. A *connection* on M is a g-connection on M with $\mathsf{g} = \mathrm{Der}_k(A)$.

Let ∇ be a g-connection on M. We define the *curvature* of ∇ to be the A-linear map $R_\nabla : \mathsf{g} \wedge \mathsf{g} \to \mathrm{End}_A(M)$ given by

$$R_\nabla(D \wedge D') = [\nabla_D, \nabla_{D'}] - \nabla_{[D,D']}$$

for all $D, D' \in \mathsf{g}$. We say that ∇ is an *integrable* g*-connection* if $R_\nabla = 0$.

We define $\mathsf{MC}(A, \mathsf{g})$ to be the category of *modules with* g*-connections*. The objects in $\mathsf{MC}(A, \mathsf{g})$ are pairs (M, ∇), where M is an A-module and ∇ is a g-connection on M, and the morphisms $\phi : (M, \nabla) \to (M', \nabla')$ in $\mathsf{MC}(A, \mathsf{g})$ are the *horizontal maps*, i.e. A-linear homomorphisms $\phi : M \to M'$ such that $\phi\nabla_D = \nabla'_D\phi$ for all $D \in \mathsf{g}$. The category $\mathsf{MC}(A, \mathsf{g})$ is an Abelian k-category, and we write $\mathsf{MC}(A) = \mathsf{MC}(A, \mathsf{g})$ when $\mathsf{g} = \mathrm{Der}_k(A)$.

Let $\mathsf{MIC}(A, \mathsf{g})$ be the full subcategory of $\mathsf{MC}(A, \mathsf{g})$ of *modules with integrable* g*-connections*. This is an Abelian subcategory with many nice properties. In fact, there is an associative k-algebra $\Delta(A, \mathsf{g})$ such that the category $\mathsf{MIC}(A, \mathsf{g})$ is equivalent to the category of left modules over $\Delta(A, \mathsf{g})$. When $\mathsf{g} \subseteq \mathrm{Der}_k(A)$, $\Delta(A, \mathsf{g})$ is the subalgebra of $\mathsf{Diff}(A)$ generated by A and g, where $\mathsf{Diff}(A)$ denotes the ring of differential operators on A in the sense of Grothendieck [17]. When $\mathsf{g} = \mathrm{Der}_k(A)$, the algebra $\Delta(A) = \Delta(A, \mathsf{g})$ is called the *derivation algebra*.

We recall that when A is a regular k-algebra, a *connection* on M is usually defined as a k-linear map $\nabla : M \to M \otimes_A \Omega_A$ such that $\nabla(am) = a\nabla(m) + m \otimes d(a)$ for all $a \in A,\ m \in M$, see Katz [25]. Moreover, the *curvature* of ∇ is usually defined as the A-linear map $R_\nabla : M \to M \otimes_A \Omega^2_A$ given by $R_\nabla = \nabla^1 \circ \nabla$, where ∇^1 is the natural extension of ∇ to $M \otimes \Omega_A$, and ∇ is an *integrable connection* if $R_\nabla = 0$.

Let A be any commutative k-algebra. For expository purposes, we define an Ω*-connection* on an A-module M to be a connection on M in the sense of the preceding paragraph. We define $\Omega\mathsf{MC}(A)$ to be the Abelian k-category of modules

with Ω-connections, and $\Omega\mathsf{MIC}(A)$ to be the full Abelian subcategory of modules with integrable Ω-connections.

Lemma 1. *Let A be a regular local k-algebra essentially of finite type, and let M be a finitely generated A-module. If there is an Ω-connection on M, then M is free.*

Proof. Let $\{t_1, \dots, t_d\}$ be a set of regular parameters of A, and let δ_i be the derivation on $k[t_1, \dots, t_d]$ such that $\delta_i(t_j) = \delta_{ij}$ for $1 \leq i \leq d$. If A is essentially of finite type over k, then δ_i extends to a derivation of A. This implies that any module M that admits an Ω-connection is free, see for instance Borel et al. [5], section VI, proposition 1.7. □

Lemma 2. *There is a natural functor $\Omega\mathsf{MC}(A) \to \mathsf{MC}(A)$, and an induced functor $\Omega\mathsf{MIC}(A) \to \mathsf{MIC}(A)$. If Ω_A and $\mathrm{Der}_k(A)$ are projective A-modules of finite presentation, then these functors are equivalences of categories.*

Proof. Any Ω-connection on M induces a connection on M, and this assignment preserves integrability. Moreover, any connection ∇ on M may be considered as a k-linear map $M \to \mathrm{Hom}_A(\mathrm{Der}_k(A), M)$, given by $m \mapsto \{D \mapsto \nabla_D(m)\}$. It is sufficient to show that the natural map $M \otimes_A \Omega_A \to \mathrm{Hom}_A(\mathrm{Der}_k(A), M)$, given by $m \otimes \omega \mapsto \{D \mapsto \phi_D(\omega)m\}$, is an isomorphism. But this is clearly the case when Ω_A and $\mathrm{Der}_k(A)$ are projective A-modules of finite presentation. □

We see that if A is a regular k-algebra essentially of finite type, then there is a bijective correspondence between (integrable) connections on M and (integrable) Ω-connections on M for any A-module M. In contrast, there are many modules that admit connections but not Ω-connections when A is a singular k-algebra.

When A is a complete local k-algebra, we must replace the tensor and wedge products with their formal analogues. We remark that with this modification, lemma 1 and lemma 2 hold for any complete local Noetherian k-algebra A with residue field k.

2. Elementary properties of connections

Let k be an algebraically closed field of characteristic 0, let A be a commutative k-algebra, and let g be a Lie-Rinehart algebra. If (M, ∇) and (M', ∇') are modules with g-connections, then there are natural induced g-connections on the A-modules $M \oplus M'$ and $\mathrm{Hom}_A(M, M')$. These g-connections are integrable if and only if ∇ and ∇' are integrable g-connections.

Lemma 3. *For any A-modules M, M', $M \oplus M'$ admits a g-connection if and only if M and M' admit g-connections.*

Lemma 4. *For any reflexive A-module M, $M^\vee = \mathrm{Hom}_A(M, A)$ admits a g-connection if and only if M admits a g-connection.*

In view of lemma 1 and lemma 2, we expect that any A-module that admits a connection must be locally free outside the singular locus of $\mathrm{Spec}(A)$. In order to prove this, we need some results on localizations of connections.

Let $A \to S^{-1}A$ be the localization given by a multiplicatively closed subset $S \subseteq A$. Since any derivation of A can be extended to a derivation of $S^{-1}A$, we see that $S^{-1}\mathsf{g}$ is a Lie-Rinehart algebra of $S^{-1}A/k$.

Lemma 5. *Localization gives a functor* $\mathsf{MC}(A, \mathsf{g}) \to \mathsf{MC}(S^{-1}A, S^{-1}\mathsf{g})$ *and an induced functor* $\mathsf{MIC}(A, \mathsf{g}) \to \mathsf{MIC}(S^{-1}A, S^{-1}\mathsf{g})$ *for any multiplicatively closed subset* $S \subseteq A$*.*

Let $A \to \widehat{A}$ be the m-adic completion of A given by a maximal ideal $m \subseteq A$. Since any derivation of A can be extended to a derivation of $\widehat{A}$, we see that $\widehat{A} \otimes_A \mathsf{g}$ is a Lie-Rinehart algebra of $\widehat{A}/k$. Moreover, if A is Noetherian and g is a finitely generated A-module, then $\widehat{A} \otimes_A \mathsf{g} \cong \widehat{\mathsf{g}}$.

Lemma 6. *If* A *is a Noetherian* k*-algebra and* g *is a finitely generated* A*-module,* m*-adic completion gives a functor* $\mathsf{MC}(A, \mathsf{g}) \to \mathsf{MC}(\widehat{A}, \widehat{\mathsf{g}})$ *and an induced functor* $\mathsf{MIC}(A, \mathsf{g}) \to \mathsf{MIC}(\widehat{A}, \widehat{\mathsf{g}})$ *for any maximal ideal* $m \subseteq A$*.*

In particular, if A is essentially of finite type over k, then there are localization functors $\mathsf{MC}(A) \to \mathsf{MC}(S^{-1}A)$ and $\mathsf{MIC}(A) \to \mathsf{MIC}(S^{-1}A)$ for any multiplicatively closed subset $S \subseteq A$, and m-adic completion functors $\mathsf{MC}(A) \to \mathsf{MC}(\widehat{A})$ and $\mathsf{MIC}(A) \to \mathsf{MIC}(\widehat{A})$ for any maximal ideal $m \subseteq A$.

Lemma 7. *Let* A *be a* k*-algebra essentially of finite type, and let* M *be a finitely generated* A*-module. If there is a connection on* M*, then* M_p *is a locally free* A_p*-module for all prime ideals* $p \subseteq A$ *such that* A_p *is a regular local ring.*

This lemma also holds when A is a complete local Noetherian k-algebra with residue field k. We remark that lemma 7 gives a necessary condition for a module to admit connections, and it is well-known that maximal Cohen-Macaulay modules satisfy this condition.

Lemma 8. *Let* A *be a* k*-algebra essentially of finite type, let* g *be a Lie-Rinehart algebra of* A/k *that is finitely generated as an* A*-module, and let* M *be a finitely generated* A*-module. For any maximal ideal* $m \subseteq A$*, we write* $\widehat{\mathsf{g}}$ *and* $\widehat{M}$ *for the* m*-adic completions of* g *and* M*, and consider the following statements:*

1. M *admits a* g*-connection*
2. M_m *admits a* g_m*-connection*
3. $\widehat{M}$ *admits a* $\widehat{\mathsf{g}}$*-connection*

Then we have $(1) \Rightarrow (2) \Leftrightarrow (3)$*. Moreover, if* M_p *admits a* g_p*-connection for all prime ideals* $p \neq m$ *in* A*, then* $(1) \Leftrightarrow (2)$*.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ is a direct consequence of lemma 5 and lemma 6. By the obstruction theory for connections, see Eriksen, Gustavsen [11], it follows that $(2) \Leftrightarrow (3)$. Furthermore, if there is a g_p-connection on M_p for all prime ideals $p \neq m$, it also follows that (3) implies (1). □

3. Graded connections

Let k be an algebraically closed field of characteristic 0, and let A be a *quasi-homogeneous* k-algebra, i.e. a positively graded k-algebra of the form $A \cong S/I$, where $S = k[x_1, \dots, x_n]$ is a graded polynomial ring with $\deg(x_i) > 0$ for $1 \leq i \leq n$, and I is a homogeneous ideal in S.

We see that $\mathrm{Der}_k(A)$ has a natural grading induced by the grading of A such that the homogeneous derivations $D \in \mathrm{Der}_k(A)$ of degree ω satisfy $D(A_i) \subseteq A_{i+\omega}$ for all integers $i \geq 0$. We also notice that $\mathrm{Der}_k(A)$ is a graded k-Lie algebra, i.e. $[\mathrm{Der}_k(A)_i, \mathrm{Der}_k(A)_j] \subseteq \mathrm{Der}_k(A)_{i+j}$ for all integers i, j.

We say that a Lie-Rinehart algebra (g, τ) is *graded* if g is a graded A-module and k-Lie algebra, and $\tau : \mathsf{g} \to \mathrm{Der}_k(A)$ is a graded homomorphism of A-modules and k-Lie algebras.

Let g be a graded Lie-Rinehart algebra. For any graded A-module M, we define a *graded* g*-connection* on M to be a g-connection ∇ on M such that $\nabla_D(M_i) \subseteq M_{i+\omega}$ for any integer i and for any homogeneous element $D \in \mathsf{g}$ of degree ω.

Lemma 9. *Let A be a quasi-homogeneous k-algebra, let g be a graded Lie-Rinehart algebra of A/k that is finitely generated as an A-module, and let m be the graded maximal ideal of A. For any finitely generated graded A-module M, the following conditions are equivalent:*

1. *M admits a graded g-connection*
2. *M_m admits a g_m-connection*

Proof. The functor $M \mapsto M_m$ is faithfully exact on the category of finitely generated graded A-modules since A has a unique graded maximal ideal m. Hence the result follows as in lemma 8. □

Let A be a quasi-homogeneous k-algebra, and let M be a finitely generated graded A-module. We consider a graded presentation of M of the form

$$0 \leftarrow M \xleftarrow{\rho} L_0 \xleftarrow{d_0} L_1, \tag{2}$$

where L_0 and L_1 are free graded A-modules of finite rank, with homogeneous A-linear bases $\{e_i\}$ and $\{f_i\}$ respectively, and d_0 is a graded A-linear homomorphism. If we write (a_{ij}) for the matrix of d_0 with respect to the chosen bases, then a_{ij} is homogeneous with $\deg(a_{ij}) = \deg(e_i) - \deg(f_j)$ for $1 \leq i \leq \mathrm{rk}(L_0),\ 1 \leq j \leq \mathrm{rk}(L_1)$.

Let $D \in \mathrm{Der}_k(A)$ be a homogeneous derivation of degree ω. Then there is a natural action of D on L_0 and L_1, i.e. $D(ae_i) = D(a)e_i$ and $D(af_j) = D(a)f_j$ for any $a \in A$ and any $e_i \in L_0,\ f_j \in L_1$. For simplicity, we shall denote the induced k-linear endomorphisms by $D : L_n \to L_n$ for $n = 0, 1$, and write $D(d_0) : L_1 \to L_0$ for the A-linear homomorphism given by $D(d_0) = Dd_0 - d_0D$. Notice that $D : L_n \to L_n$ is graded of degree ω, hence $D(d_0)$ is also graded of degree ω.

Lemma 10. *Let $D \in \mathrm{Der}_k(A)_w$ for some integer ω, and let $\nabla_D \in \mathrm{End}_k(M)$ be a k-linear endomorphism with derivation property with respect to D such that $\nabla_D(M_i) \subseteq M_{i+\omega}$ for all integers i. Then there exist A-linear endomorphisms $\phi_D \in \mathrm{End}_A(L_0)_\omega$ and $\psi_D \in \mathrm{End}_A(L_1)_\omega$ with $D(d_0) = d_0\psi_D - \phi_D d_0$ such that ∇_D is induced by $D + \phi_D : L_0 \to L_0$.*

Proof. Consider the map $\nabla_D\,\rho - \rho D : L_0 \to M$, and notice that it is a graded A-linear homomorphism of degree ω. Hence we can find a graded A-linear endomorphism $\phi_D : L_0 \to L_0$ of degree ω such that $\nabla_D\,\rho = \rho(D + \phi_D)$, and this implies that $(D + \phi_D)d_0(x) \in \mathrm{im}(d_0)$ for all $x \in L_1$. So we can find a graded A-linear homomorphism $\psi_D : L_1 \to L_1$ of degree ω such that $D(d_0) = d_0\psi_D - \phi_D d_0$. □

If we write (p_{ij}) for the matrix of ϕ_D and (c_{ij}) for the matrix of ψ_D with respect to the chosen bases, then p_{ij} is homogeneous with $\deg(p_{ij}) = \deg(e_j) - \deg(e_i) + \omega$ for $1 \leq i, j \leq \mathrm{rk}(L_0)$ and c_{ij} is homogeneous with $\deg(c_{ij}) = \deg(f_j) - \deg(f_i) + \omega$ for $1 \leq i, j \leq \mathrm{rk}(L_1)$.

4. Maximal Cohen-Macaulay modules

Let k be an algebraically closed field of characteristic 0, and let A be a complete local Noetherian k-algebra with residue field k. We say that a finitely generated A-module M is *maximal Cohen-Macaulay* (MCM) if $\operatorname{depth}(M) = \dim(A)$, and that A is a *Cohen-Macaulay* (CM) ring if A is MCM as an A-module. In the rest of this paper, we shall assume that A is a CM ring and an isolated singularity.

We say that A has *finite CM representation type* if the number of isomorphism classes of indecomposable MCM A-modules is finite. The singularities of finite CM representation type has been classified when A has dimension $d \leq 2$, but not in higher dimensions.

Let A be the complete local ring of a *simple hypersurface singularity*, see Arnold [1] and Wall [30]. Then $A \cong k[[z_0, \dots, z_d]]/(f)$, where $d \geq 1$ is the dimension of A and f is of the form

$$\begin{aligned}
&A_n : f = z_0^2 + z_1^{n+1} + z_2^2 + \cdots + z_d^2 && n \geq 1\\
&D_n : f = z_0^2 z_1 + z_1^{n-1} + z_2^2 + \cdots + z_d^2 && n \geq 4\\
&E_6 : f = z_0^3 + z_1^4 + z_2^2 + \cdots + z_d^2\\
&E_7 : f = z_0^3 + z_0 z_1^3 + z_2^2 + \cdots + z_d^2\\
&E_8 : f = z_0^3 + z_1^5 + z_2^2 + \cdots + z_d^2
\end{aligned}$$

The simple singularities are exactly the hypersurface singularities of finite CM representation type, see Knörrer [26] and Buchweitz, Greuel, Schreyer [7]. Moreover, if A is Gorenstein and of finite CM representation type, then A is a simple singularity, see Herzog [20].

Assume that A is a hypersurface $A = S/(f)$, where S is a power series k-algebra. A *matrix factorization* of f is a pair (ϕ, ψ) of square matrices with entries in S such that $\phi\psi = \psi\phi = I$. We say that (ϕ, ψ) is a *reduced matrix factorization* if the entries in ϕ and ψ are non-units in S. By Eisenbud [9], there is a bijective correspondence between reduced matrix factorizations of f and MCM A-modules without free summands, given by the assignment $(\phi, \psi) \mapsto \operatorname{coker}(\phi)$.

Let $f \in S = k[[z_0, \dots, z_d]]$ be the equation of the simple singularity $A = S/(f)$ of dimension d, and let (ϕ, ψ) be a reduced matrix factorization of $f \in S$. Then the pair (ϕ', ψ') given by

$$\phi' = \begin{pmatrix} uI & -\psi \\ \phi & vI \end{pmatrix}, \psi' = \begin{pmatrix} vI & \psi \\ -\phi & uI \end{pmatrix}$$

is a reduced matrix factorization of $f' = f + uv \in S' = S[[u, v]]$, and $A' = S'/(f')$ is isomorphic to the simple singularity of dimension $d + 2$ corresponding to A. By Knörrer's periodicity theorem, this assignment induces a bijective correspondence between MCM A-modules without free summands and MCM A'-modules without free summands, see Knörrer [26] and Schreyer [29].

A complete list of indecomposable MCM modules over simple curve singularities was given in Greuel, Knörrer [15], and the corresponding matrix factorizations were given in Eriksen [13] and Yoshino [31]. A complete list of indecomposable MCM modules over simple surface singularities can be obtained from the irreducible representations of the finite subgroups of $\mathrm{SL}(2, k)$. In higher dimensions, a complete list of indecomposable MCM modules over simple singularities can be obtained using Knörrer's periodicity theorem.

Lemma 11. *Let (ϕ, ψ) be a reduced matrix factorization of $f \in S$, let $A = S/(f)$, and let $M = \mathrm{coker}(\phi)$. Then we have:*

1. $\mathrm{coker}(\phi^t) \cong M^\vee$*, the A-linear dual of M,*
2. $\mathrm{coker}(\psi) \cong \mathrm{syz}^1(M)$*, the first reduced syzygy of M.*

We remark that by lemma 4, there is a connection on M if and only if there is a connection on $M^\vee$. It is not difficult to prove that there is a k-linear connection on M if and only if there is a k-linear connection on $\mathrm{syz}^1(M)$ using the above result, and a different proof of this fact was given in Källström [23], theorem 2.2.8. However, we do not know if is true that there is a connection on M if and only if there is a connection of $\mathrm{syz}^1(M)$.

5. Connections on MCM modules

Let k be an algebraically closed field of characteristic 0, and let A be a complete local CM k-algebra with residue field k that is an isolated singularity. In this section, we study the existence of connections on MCM A-modules under these conditions. We focus on the cases when A has finite CM representation type.

5.1. **Dimension zero.** When A is a complete local ring of a zero-dimensional singularity, it has finite CM representation type if and only if it is of the form $A = k[[x]]/(x^{n+1})$ for some integer $n \geq 1$, see Herzog [20], satz 1.5. We consider these singularities as the zero-dimensional simple singularities of type A_n.

We remark that there are $n - 1$ reduced matrix factorizations of x^n, given by $x^n = x^i \cdot x^{n-i}$ for $1 \leq i \leq n - 1$. Since the natural action of the Euler derivation $E = x\frac{\partial}{\partial x}$ on x^i is given by $E(x^i) = ix^i$, we see that any MCM A-module admits an integrable connection.

5.2. **Dimension one.** When A is the complete local ring of a simple curve singularity, it was shown in Eriksen [13] that any MCM A-module admits an integrable connection.

Let A be the complete local ring of a curve singularity. We say that a local ring B *birationally dominates* A if $A \subseteq B \subseteq A^*$, where A^* is the integral closure of A in its total quotient ring. It is known that A has finite CM representation type if and only if it birationally dominates the complete local ring of a simple curve singularity, see Greuel, Knörrer [15].

This result leads to a complete classification of curve singularities of finite CM representation type. The non-Gorenstein curve singularities of finite CM representation type are of the following form:

$$
\begin{aligned}
D_n^s &: A = k[[x,y,z]]/(x^2 - y^n, xz, yz) \text{ for } n \geq 2 \\
E_6^s &: A = k[[t^3, t^4, t^5]] \subseteq k[[t]] \\
E_7^s &: A = k[[x,y,z]]/(x^3 - y^4, xz - y^2, y^2z - x^2, yz^2 - xy) \\
E_8^s &: A = k[[t^3, t^5, t^7]] \subseteq k[[t]]
\end{aligned}
$$

Using Singular [16] and our library conn.lib [10], we show that not all MCM A-modules admit connections in these cases. In fact, the canonical module ω_A does not admit a connection when A is the complete local ring of the singularities E_6^s, E_7^s, E_8^s or D_n^s for $n \leq 100$.

The *monomial curve singularities* is an interesting class of curve singularities not necessarily of finite CM representation type. Let $\Gamma \subseteq \mathbf{N_0}$ be a *numerical semigroup*,

i.e. a sub-semigroup $\Gamma \subseteq \mathbf{N_0}$ such that the complement $H = \mathbf{N_0} \setminus \Gamma$ is finite, and let $A = k[[\Gamma]]$ be the complete local ring of the corresponding monomial curve singularity, i.e. the subalgebra $k[[t^{a_1}, t^{a_2}, \dots, t^{a_r}]] \subseteq k[[t]]$, where $\{a_1, a_2, \dots, a_r\}$ is the minimal set of generators of Γ. Let the *Frobenius number* g of Γ be the maximal element of H. It is well-known that A is Gorenstein if and only if Γ is *symmetric*, i.e. $a \in \Gamma$ if and only if $g - a \notin \Gamma$ for any integer $a \in \mathbf{Z}$. If Γ is not symmetric, we denote by Δ the non-empty set $\Delta = \{h \in H : g - h \in H\}$.

Let Λ be a set such that $\Gamma \subseteq \Lambda \subseteq \mathbf{N_0}$ and $\Gamma + \Lambda \subseteq \Lambda$, and consider the module $M = k[[\Lambda]]$ with k-linear basis $\{t^a : a \in \Lambda\}$ and the obvious action of A. It is clear that $M = k[[\Lambda]]$ is an MCM A-module of rank one, and that $M = k[[\Lambda]]$ is isomorphic to $M' = k[[\Lambda']]$ if and only if $\Lambda = \Lambda'$. In fact, one may show that the rank one MCM modules over $A = k[[\Gamma]]$ of the form $M = k[[\Lambda]]$ are exactly the *gradable* ones, i.e. the A-modules induced by graded modules over the quasi-homogeneous algebra $k[\Gamma] = k[t^{a_1}, t^{a_2}, \dots, t^{a_r}] \subseteq k[t]$.

Proposition 12. *Let $A = k[[\Gamma]]$ be a monomial curve singularity, let Λ be set such that $\Gamma \subseteq \Lambda \subseteq \mathbf{N_0}$ and $\Gamma + \Lambda \subseteq \Lambda$, and let $M = k[[\Lambda]]$ be the corresponding rank one MCM A-module. If $g - s \in \Lambda$ and $g \notin \Lambda$, where $s = \max \Delta$, then M does not admit connections.*

Proof. We see that $E = t\frac{\partial}{\partial t}$, $D = t^g E$ and $D' = t^s E$ are derivations of A, see Eriksen [12], the remarks preceding lemma 8. If ∇ is a connection on M, then there exists an element $f = f_0 + f_+ \in k[[\Lambda]]$ with $f_0 \in k$ and $f_+ \in (t)$ such that $\nabla_E(t^\lambda) = E(t^\lambda) + f t^\lambda$ for all $\lambda \in \Lambda$. Since M is torsion free, we must have $\nabla_D(t^\lambda) = D(t^\lambda) + t^g f t^\lambda \in M$ for all $\lambda \in \Lambda$. For $\lambda = 0$, this condition implies that $f_0 = 0$. Similarly, we must have $\nabla_{D'}(t^\lambda) = D'(t^\lambda) + t^s f t^\lambda \in M$ for all $\lambda \in \Lambda$. For $\lambda = g - s$, this condition implies that $(g - s) + f_0 = 0$, which is a contradiction. □

Theorem 13. *Let A be the complete local ring of a monomial curve singularity. Then all gradable MCM A-module of rank one admits a connection if and only if A is Gorenstein.*

Proof. Let $A = k[[\Gamma]]$ for a numerical semigroup Γ, and $M = k[[\Lambda]]$ for a set Λ with $\Gamma \subseteq \Lambda \subseteq \mathbf{N_0}$ and $\Gamma + \Lambda \subseteq \Lambda$. If A is Gorenstein, then $\mathrm{Der}_k(A)$ is generated by the Euler derivation $E = t\frac{\partial}{\partial t}$ and the trivial derivation $D = t^g E$, see Eriksen [12], lemma 8 and the following remarks. Hence the natural action of E and D on M induces a connection on M since $g + (\Lambda \setminus \{0\}) \subseteq \Gamma$ and $D(1) = 0$. On the other hand, if A is not Gorenstein, then the set Δ is non-empty, and $\Lambda = \Gamma \cup (g - s + \Gamma)$ satisfies the conditions $\Gamma \subseteq \Lambda \subseteq \mathbf{N_0}$ and $\Gamma + \Lambda \subseteq \Lambda$. Since $g \notin \Lambda$, it follows from proposition 12 that M does not admit connections. □

Let us consider the non-Gorenstein monomial curve singularities E_6^s and E_8^s of finite CM representation type. By Yoshino [31], theorem 15.14, all MCM A-modules are gradable in these cases. For E_6^s, we have $H = \{1, 2\}$, and the possibilities for Λ (with $\Lambda \neq \Gamma$) are

$$\Lambda_1 = \Gamma \cup \{1\}, \quad \Lambda_2 = \Gamma \cup \{2\}, \quad \Lambda_{12} = \Gamma \cup \{1, 2\}$$

The corresponding modules M_1, M_2, M_{12} are the non-free rank one MCM A-modules. Using the method from the proof of proposition 12, we see that the modules M_2 and M_{12} admit connections, while M_1 does not. One may show that M_1 is the canonical module in this case. A similar consideration for E_8^s shows that M_{14},

M_2, M_4, M_{24} and M_{124} are the non-free rank one MCM A-modules, and that the canonical module M_2 is the only rank one MCM A-module that does not admit connections.

Let us also consider a non-Gorenstein monomial curve singularity not of finite CM representation type. When $A = k[[t^4, t^5, t^6, t^7]]$, we have $H = \{1, 2, 3\}$, and the possibilities for Λ (with $\Lambda \neq \Gamma$) are

$$\Lambda_1 = \Gamma \cup \{1\}, \quad \Lambda_2 = \Gamma \cup \{2\}, \quad \Lambda_3 = \Gamma \cup \{3\}, \quad \Lambda_{12} = \Gamma \cup \{1, 2\},$$
$$\Lambda_{13} = \Gamma \cup \{1, 3\}, \quad \Lambda_{23} = \Gamma \cup \{2, 3\}, \quad \Lambda_{123} = \Gamma \cup \{1, 2, 3\}$$

The corresponding modules $M_1, M_2, M_3, M_{12}, M_{13}, M_{23}$ and M_{123} are the non-free rank one gradable MCM A-modules. One may show that the modules M_3, M_{23} and M_{123} admit connections, that the canonical module M_{12} and the module M_{13} admit k-linear connections but not connections, while the modules M_1 and M_2 do not even admit k-linear connections.

Finally, we remark that any connection on an MCM A-module is integrable when A is a monomial curve singularity.

5.3. **Dimension two.** When A is the complete local ring of a surface singularity, it has finite CM representation type if and only if it is a quotient singularity of the form $A = S^G$, where $S = k[[x, y]]$ and G is a finite subgroup of $\mathrm{GL}(2, k)$ without pseudo-reflections. This fact was proven independently in Auslander [3] and Esnault [14]. Moreover, there is a bijective correspondence between MCM A-modules and finite dimensional representations of G.

It is not difficult to see that any MCM module over a quotient surface singularity admits an integrable connection, and Jan Christophersen was the first to point this out to us.

Proposition 14. *Let A be the complete local ring of a quotient singularity $A = S^G$, where $S = k[[x_1, \dots, x_n]]$ and $G \subseteq \mathrm{GL}(n, k)$ is a finite subgroup without pseudo-reflections. For any finite dimensional representation $\rho : G \to \mathrm{End}_k(V)$, the MCM A-module $M = (S \otimes_k V)^G$ admits an integrable connection.*

Proof. There is a canonical integrable connection $\nabla' : \mathrm{Der}_k(S) \to \mathrm{End}_k(S \otimes_k V)$ on the free S-module $S \otimes_k V$, given by

$$\nabla'_D(\sum s_i \otimes v_i) = \sum D(s_i) \otimes v_i$$

for any $D \in \mathrm{Der}_k(S),\ s_i \in S,\ v_i \in V$. But the natural map $\mathrm{Der}_k(S)^G \to \mathrm{Der}_k(S^G)$ is an isomorphism, see Kantor [24] or Schlessinger [28], hence ∇' induces an integrable connection ∇ on M. □

Theorem 15. *Let A be the complete local ring of a surface singularity of finite CM representation type. Then any MCM A-module admits an integrable connection.*

Proof. By the comments preceding proposition 14, we may assume that $A = S^G$, where $S = k[[x, y]]$ and G is a finite subgroup of $\mathrm{GL}(2, k)$ without pseudo-reflections, and that $M = (S \otimes_k V)^G$ for a finite dimensional representation $\rho : G \to \mathrm{End}_k(V)$ of G. Hence M admits an integrable connection by proposition 14. □

We recall that the simple surface singularities are precisely the quotient surface singularities that are Gorenstein. We may also characterize them as the quotient surface singularities $A = S^G$ where G is a subgroup of $\mathrm{SL}(2, k)$. In particular, we

see that any MCM A-module over a simple surface singularity admits an integrable connection.

Let A be the complete local ring of any surface singularity. We remark that theorem 15 can be generalized as follows: Let us consider a finite Galois extension L of K, where K is the field of fractions of A, and let B be the integral closure of A in L. If the extension $A \subseteq B$ is unramified at all height one prime ideals, we say that it is a *Galois extension*. In Gustavsen, Ile [19], it was proven that if M is an MCM A-module such that $(M \otimes_A B)^{\vee\vee}$ is a free B-module for some Galois extension $A \subseteq B$, then M admits an integrable connection. In particular, if A is a rational surface singularity, it follows that any rank one MCM A-module admits an integrable connection.

If $A = S^G$ is a quotient singularity, then $A \subseteq S = k[[x, y]]$ is a Galois extension such that $(M \otimes_A S)^{\vee\vee}$ is free for any MCM A-module M. In contrast, if A is not a quotient singularity, there exists an MCM A-module M such that $(M \otimes_A B)^{\vee\vee}$ is non-free for any Galois extension $A \subseteq B$, see Gustavsen, Ile [19]. Nevertheless, M may still admit an integrable connection. In fact, any MCM module over a simple elliptic surface singularity admits an integrable connection, see Kahn [22], and this result has been generalized to quotients of simple elliptic surface singularities in Gustavsen, Ile [18]. Kurt Behnke has pointed out that it might be true for cusp singularities as well, see Behnke [4]. More generally, it is probable that any MCM module over a log canonical surface singularity admits an integrable connection.

When A is a surface singularity, we have not found any examples of an MCM module that does not admit an integrable connection.

5.4. **Higher dimensions.** The main result of this paper is that when A is the complete local ring of a simple singularity of dimension $d \geq 3$, an MCM A-module M admits a connection only if M is free. In Eriksen, Gustavsen [11], we used Singular [16] and our library conn.lib [10] to prove this result when A is a simple threefold singularity of type A_n, D_n or E_n with $n \leq 50$. Using graded techniques, we are able to prove this result for any simple singularity of dimension $d \geq 3$:

Theorem 16. *Let A be the complete local ring of a simple singularity of dimension $d \geq 3$. Then an MCM A-module M admits a connection if and only if M is free.*

Proof. We claim that when A is a simple threefold singularity, g is the submodule of $\mathrm{Der}_k(A)$ generated by the trivial derivations, and M is a non-free MCM A-module, then M does not admit g-connections.

Let us first prove that the claim implies the theorem. If A is a simple threefold singularity and M is a non-free MCM A-module that admits a connection, then its restriction to g is a g-connection on M. So we may assume that A has dimension $d > 3$. Assume that there is a non-free MCM A-module M that admits a connection. Clearly, $A = S[[u_1, \dots, u_{d-3}]]/(f)$ with $f = f_0 + u_1^2 + \dots + u_{d-3}^2$, where $S/(f_0)$ is the simple threefold singularity of the same type as A. Moreover, M is given by a reduced matrix factorization (ϕ, ψ) of f. Let us write g for the submodule of $\mathrm{Der}_k(S/(f_0))$ generated by the trivial derivations. Since $A/(u_1, \dots, u_{d-3}) \cong S/(f_0)$, $M_0 = M/(u_1, \dots, u_{d-3})M$ is a non-free MCM $S/(f_0)$-module, given by the reduced matrix factorization $(\phi \otimes_A A/(u_1, \dots, u_{d-3}), \psi \otimes_A A/(u_1, \dots, u_{d-3}))$ of f_0. Since M admits a connection, its restriction to $\mathsf{g}_A = A \cdot \mathsf{g} \subseteq \mathrm{Der}_k(A)$, the A-submodule of $\mathrm{Der}_k(A)$ generated by the trivial derivations of $S/(f)$, is a g_A-connection on M.

Since any g_A-connection on M induces a g-connection on M_0, this contradicts the claim.

It remains to prove the claim. Let $A' = k[[z_0, z_1, z_2, z_3]]/(f)$ be the complete local ring of a simple threefold singularity, and let M' be a non-free MCM A'-module. We shall show that M' does not admit a g-connection. Calculations in SINGULAR, mentioned in the comments preceding the theorem, shows that the claim is true when A' is of type E_6, E_7 and E_8, so we may assume that A' is of type A_n or D_n. Moreover, we may assume that M' is indecomposable by lemma 3.

Let $A = k[z_0, z_1, z_2, z_3]/(f)$ be the quasi-homogeneous k-algebra that corresponds to the simple threefold singularity of type A_n or D_n, i.e. $\widehat{A} \cong A'$. By Yoshino [31], theorem 15.14, we may assume that $M' \cong \widehat{M}$ for some graded MCM A-module M, and clearly M is non-free and indecomposable. Since g is generated by homogeneous derivations, it is a graded Lie-Rinehart algebra. By lemma 8 and lemma 9, it is therefore enough to show that if M is a graded indecomposable non-free MCM A-module, then M does not admit a graded g-connection.

Assume that $\nabla : \mathsf{g} \to \mathrm{End}_k(M)$ is a graded g-connection on M for some graded indecomposable non-free MCM A-module M, and let $d_0 : L_1 \to L_0$ be a graded presentation of M. For any homogeneous derivation $D \in \mathsf{g}$ of degree ω, it follows from lemma 10 that ∇_D is induced by an operator of the form $D + \phi_D : L_0 \to L_0$, where $\phi_D \in \mathrm{End}_A(L_0)_\omega$ satisfies $D(d_0) = d_0\psi_D - \phi_D d_0$ for some $\psi_D \in \mathrm{End}_A(L_1)_\omega$. Moreover, for any graded relation $\sum a_i D_i = 0$ in g, we must have $\sum a_i \phi_{D_i} = 0$ in $\mathrm{End}_A(M)$.

In the A_n case, we choose coordinates such that $A = k[x, y, z, w]/(f)$, where $f = x^2 + y^{n+1} + zw$ for some $n \geq 1$. There are graded trivial derivations D_1, D_2, D_3 of A, all of degree 0, given by

$$D_1 = z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}, \quad D_2 = w\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial z}, \quad D_3 = z\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial w}$$

and there is a graded relation $2xD_1 + zD_2 - wD_3 = 0$. For all graded indecomposable non-free MCM A-module M, this leads to a contradiction, see appendix A for details.

In the D_n case, we choose coordinates such that $A = k[x, y, z, w]/(f)$, where $f = x^2y + y^{n-1} + zw$ for some $n \geq 4$. There are graded trivial derivations D_1, D_2, D_3 of A, with D_1 of degree 0 and D_2, D_3 of degree 1, given by

$$D_1 = z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}, \quad D_2 = w\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial z}, \quad D_3 = z\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial w}$$

and there is a graded relation $2xyD_1 + zD_2 - wD_3 = 0$. For almost all graded indecomposable non-free MCM A-modules M, this leads to a contradiction, see appendix A for details.

However, in some exceptional cases, we shall use the graded trivial derivations D_4, D_5 of A, with degree $n - 3$, given by

$$D_4 = w\frac{\partial}{\partial y} - \beta\frac{\partial}{\partial z}, \quad D_5 = z\frac{\partial}{\partial y} - \beta\frac{\partial}{\partial w}$$

where $\beta = x^2 + (n-1)y^{n-2}$, and the graded relation $\beta D_1 + zD_4 - wD_5 = 0$. For the remaining graded indecomposable non-free MCM A-modules M, this leads to a contradiction, see appendix A for details. □

The classification of non-Gorenstein singularities of finite CM representation type is not known in dimension $d \geq 3$. However, partial results are given in Eisenbud, Herzog [8], Auslander, Reiten [2] and Yoshino [31].

In Auslander, Reiten [2], it was shown that there is only one quotient singularity of dimension $d \geq 3$ with finite CM representation type, the cyclic threefold quotient singularity of type $\frac{1}{2}(1,1,1)$. Its complete local ring $A = S^G$ is the invariant ring of the action of the group $G = \mathbf{Z}_2$ on $S = k[[x_1, x_2, x_3]]$ given by $\sigma x_i = -x_i$ for $i = 1, 2, 3$, where $\sigma \in G$ is the non-trivial element. There are exactly two non-free indecomposable MCM A-modules M_1 and M_2. The module M_1 has rank one, and is induced by the non-trivial representation of G of dimension one, see Auslander, Reiten [2]. By proposition 14, it follows that M_1 admits an integrable connection. One may show that M_1 is the canonical module of A.

It is also known that the threefold scroll of type $(2, 1)$, with complete local ring $A = k[[x, y, z, u, v]]/(xz-y^2, xv-yu, yv-zu)$, has finite CM representation type, see Auslander, Reiten [2]. There are four non-free indecomposable MCM A-modules, and none of these admit connections. In particular, the canonical module ω_A does not admit a connection.

To the best of our knowledge, no other examples of singularities of finite CM representation type are known in dimension $d \geq 3$.

6. The canonical module

Let k be an algebraically closed field of characteristic 0, and let A be a complete local CM k-algebra with residue field k that is an isolated singularity. Then A has a canonical module ω_A, an MCM A-module of finite injective dimension and with type $r(\omega_A) = \dim_k \operatorname{Ext}^d_A(k, \omega_A) = 1$, where $d = \dim(A)$, see Bruns, Herzog [6].

Since A is CM, ω_A is also a dualizing module, see Bruns, Herzog [6], theorem 3.3.10. In particular, $\operatorname{Ext}^1_A(\omega_A, \omega_A) = 0$, and it follows from the obstruction theory for connections that ω_A admits a k-linear connection, see Eriksen, Gustavsen [11]. If A is Gorenstein, then ω_A is free, hence ω_A admits an integrable connection. However, it turns out that this is not true in general, and we ask the following question: *When does the canonical module admit a connection?*

When A has dimension $d \geq 2$, it follows from our assumptions that A is normal. In this case, we say that A is $\mathbf{Q}$-*Gorenstein* if

$$\omega_A^{[n]} = (\omega_A \otimes_A \cdots \otimes_A \omega_A)^{\vee\vee} \cong A$$

for some integer $n \geq 1$. We remark that A is $\mathbf{Q}$-Gorenstein if and only if A is of the form $A = S^G$, where S is the complete local Gorenstein k-algebra with residue field k and G is a finite subgroup of $\operatorname{Aut}_k(S)$ that acts freely outside the closed point of $\operatorname{Spec}(S)$.

Theorem 17. *Let A be the complete local ring of a singularity of dimension $d \geq 1$. If A is a Gorenstein curve or a $\mathbf{Q}$-Gorenstein singularity of dimension $d \geq 2$, then the canonical module ω_A admits an integrable connection.*

Proof. If A is a Gorenstein curve, then $\omega_A \cong A$ and the result is trivial. We may therefore assume that $A = S^G$ for a Gorenstein singularity S and a finite subgroup G of $\operatorname{Aut}_k(S)$ that acts freely outside the closed point of $\operatorname{Spec}(S)$. This implies that there is a character χ of G such that ω_A is isomorphic to the semi-invariants $S^\chi = \{a \in S : ga = \chi(g)a \text{ for all } g \in G\}$, see Hinič [21]. Since A is an isolated

singularity of dimension at least two, the canonical map $\mathrm{Der}_k(S)^G \to \mathrm{Der}_k(S^G)$ is an isomorphism, see Schlessinger [28]. Hence it follows from the proof of proposition 14 that ω_A admits an integrable connection. □

We remark that it is *not true* that the canonical module ω_A admits an integrable connection when A is a quotient of the form $A = S^G$, where S is a Gorenstein curve singularity and G is a finite subgroup of $\mathrm{Aut}_k(S)$ that acts freely outside the closed point of $\mathrm{Spec}(S)$. In fact, consider the Gorenstein monomial curve singularity $S = k[[t^3, t^5]]$, let $G = \mathbf{Z}_2$, and let the non-trivial element $\sigma \in \mathbf{Z}_2$ act on S as follows:

$$\sigma t^3 = -t^3, \quad \sigma t^5 = -t^5$$

Then $A = S^G = k[[t^6, t^8, t^{10}]] \cong k[[t^3, t^4, t^5]]$, and we see that A is non-Gorenstein. This implies that ω_A does not admit connections, as the following theorem shows:

Theorem 18. *Let A be the complete local ring of a monomial curve singularity. Then the canonical module ω_A admits a connection if and only if A is Gorenstein.*

Proof. The canonical module is characterized by its Hilbert function, and we see from the characterization in Bruns, Herzog [6], theorem 4.4.6 that the canonical module $\omega_A \cong k[[\Lambda]]$, where $\Lambda = \Gamma \cup (\Gamma + \Delta)$ (using the notation of subsection 5.2). Since $g \notin \Lambda$, the result follows from proposition 12 and theorem 13. □

We would like to find necessary conditions for ω_A to admit a connection, and the sufficient condition given in theorem 17 is a natural candidate. In dimension one, we have seen that the Gorenstein condition is necessary for any monomial curve and for the curves D_n^s for $n \leq 100$ and E_6^s, E_7^s, E_8^s. In dimension $d \geq 2$, we have seen that the $\mathbf{Q}$-Gorenstein condition is necessary for all known examples of singularities of finite CM representation type. We make the following conjecture:

Conjecture 19. *Let A be the complete local ring of a singularity of dimension $d \geq 1$. Then the canonical module ω_A admits a connection if and only if A is a Gorenstein curve or a $\mathbf{Q}$-Gorenstein singularity of dimension $d \geq 2$.*

Appendix A. Calculations for A_n and D_n in dimension 3

Let A be the quasi-homogeneous k-algebra corresponding to a simple threefold singularity of type A_n or D_n, and let $\mathsf{g} \subseteq \mathrm{Der}_k(A)$ denote the graded submodule generated by the trivial derivations. In the proof of theorem 16, we used the fact that a graded non-free indecomposable MCM A-module M does not admit a graded g-connection. There are complete lists of such modules, and for each module M in these lists, the assumption that there exists a graded g-connection on M leads to a contradiction. In this appendix, we shall give the full details of the calculations that lead to these contradictions. We use the notation of the proof of theorem 16.

A.1. The A_n case.

Let $n \geq 1$ be an integer, and consider the quasi-homogeneous k-algebra $A = k[x, y, z, w]/(f)$, where $f = x^2 + y^{n+1} + zw$ and the grading of A is given by

$$(\deg x, \deg y, \deg z, \deg w) = (n + 1, 2, n + 1, n + 1).$$

We consider the set of isomorphism classes of indecomposable graded non-free MCM A-modules (up to degree shifting).

When n is even, we may take the modules $\{M_l : 1 \leq l \leq p\}$ as representatives, where $p = \frac{n}{2}$ and the module M_l has presentation matrix

$$\mathsf{M}_l = \begin{pmatrix} z & 0 & -x & -y^{n+1-l} \\ 0 & z & -y^l & x \\ x & y^{n+1-l} & w & 0 \\ y^l & -x & 0 & w \end{pmatrix}$$

When n is odd, we may take the modules $\{M_l : 1 \leq l \leq p-1\}$ and $\{N_-, N_+\}$ as representatives, where $p = \frac{n+1}{2}$, the module M_l has presentation matrix as above, and the modules N_- and N_+ have presentation matrices

$$\mathsf{N}_- = \begin{pmatrix} z & -(x+iy^p) \\ x-iy^p & w \end{pmatrix} \qquad \mathsf{N}_+ = \begin{pmatrix} z & -(x-iy^p) \\ x+iy^p & w \end{pmatrix}$$

In each case, we view the presentation matrix as the matrix of a graded A-linear map $d_0 : L_1 \to L_0$ with respect to some chosen homogeneous bases $\{e_i\}$ for L_0 and $\{f_i\}$ for L_1, with degrees given by

$$(\deg e_1, \deg e_2, \deg e_3, \deg e_4) = (0, n+1-2l, 0, n+1-2l)$$
$$(\deg f_1, \deg f_2, \deg f_3, \deg f_4) = (n+1, 2n+2-2l, n+1, 2n+2-2l)$$

for the modules M_l, and

$$(\deg e_1, \deg e_2) = (0, 0)$$
$$(\deg f_1, \deg f_2) = (n+1, n+1)$$

for the modules N_- and N_+.

The module M_l for n even. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree 0. This implies that $\deg p_{ij} = \deg e_j - \deg e_i$ and $\deg c_{ij} = \deg f_j - \deg f_i$, so P and C must be of the form

$$P = \begin{pmatrix} p_{11} & 0 & p_{13} & 0 \\ 0 & p_{22} & 0 & p_{23} \\ p_{31} & 0 & p_{33} & 0 \\ 0 & p_{42} & 0 & p_{44} \end{pmatrix}, \; C = \begin{pmatrix} c_{11} & 0 & c_{13} & 0 \\ 0 & c_{22} & 0 & c_{23} \\ c_{31} & 0 & c_{33} & 0 \\ 0 & c_{42} & 0 & c_{44} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i, j \leq 4$. For $s = 1, 2, 3$, we must solve the equation $(D_s(\mathsf{M}_l)) = \mathsf{M}_l C_s - P_s \mathsf{M}_l$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form. This gives $P_s = P_s^0 + a_s \Psi_0$ with $a_s \in k$ for $s = 1, 2, 3$, where $\Psi_0 = I_4$ and

$$P_1^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P_2^0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, P_3^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Let $\alpha = 2xa_1 + za_2 - wa_3$. Using the relation $2xD_1 + zD_2 - wD_3 = 0$, we obtain the equation

$$2xP_1 + zP_2 - wP_3 = \begin{pmatrix} \alpha & 0 & -z & 0 \\ 0 & \alpha & 0 & z \\ -w & 0 & 2x+\alpha & 0 \\ 0 & w & 0 & 2x+\alpha \end{pmatrix} = 0$$

in $\mathrm{End}_A(M_l)$. By inspection, we see that this is a contradiction for $1 \leq l \leq p$.

The module M_l for n odd. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree 0. This implies that $\deg p_{ij} = \deg e_j - \deg e_i$ and $\deg c_{ij} = \deg f_j - \deg f_i$, so P and C must be of the form

$$P = \begin{pmatrix} p_{11} & p_{12}y^{p-l} & p_{13} & p_{14}y^{p-l} \\ 0 & p_{22} & 0 & p_{23} \\ p_{31} & p_{32}y^{p-l} & p_{33} & p_{34}y^{p-l} \\ 0 & p_{42} & 0 & p_{44} \end{pmatrix},\ C = \begin{pmatrix} c_{11} & c_{12}y^{p-l} & c_{13} & c_{14}y^{p-l} \\ 0 & c_{22} & 0 & c_{23} \\ c_{31} & c_{32}y^{p-l} & c_{33} & c_{34}y^{p-l} \\ 0 & c_{42} & 0 & c_{44} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i,j \leq 4$. For $s = 1,2,3$, we must solve the equation $(D_s(\mathsf{M}_l)) = \mathsf{M}_l C_s - P_s \mathsf{M}_l$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form. This gives $P_s = P_s^0 + a_s\Psi_0$ with $a_s \in k$ for $s = 1,2,3$, where $\Psi_0 = I_4$ and

$$P_1^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ P_2^0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ P_3^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

As in the case of the modules M_l for n even, the equation $2xP_1 + zP_2 - wP_3 = 0$ in $\mathrm{End}_A(M_l)$ leads to a contradiction for $1 \leq l \leq p-1$.

The module N_- for n odd. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree 0. This implies that $\deg p_{ij} = \deg e_j - \deg e_i$ and $\deg c_{ij} = \deg f_j - \deg f_i$, so P and C must be of the form

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},\ C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i,j \leq 2$. For $s = 1,2,3$, we must solve the equation $(D_s(\mathsf{N}_-)) = \mathsf{N}_- C_s - P_s \mathsf{N}_-$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form. This gives $P_s = P_s^0 + a_s\Psi_0$ with $a_s \in k$ for $s = 1,2,3$, where $\Psi_0 = I_2$ and

$$P_1^0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\ P_2^0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},\ P_3^0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let $\alpha = 2xa_1 + za_2 - wa_3$. Using the relation $2xD_1 + zD_2 - wD_3 = 0$, we obtain the equation

$$2xP_1 + zP_2 - wP_3 = \begin{pmatrix} \alpha & z \\ w & -2x + \alpha \end{pmatrix} = 0$$

in $\mathrm{End}_A(N_-)$. By inspection, we see that this is a contradiction.

The module N_+ for n odd. We see that $N_+ \cong N_-^\vee$ using lemma 11. It follows from lemma 4 and the computation above for N_- that the module N_+ cannot admit a g-connection.

A.2. **The D_n case.** Let $n \geq 4$ be an integer, and consider the quasi-homogeneous k-algebra $A = k[x,y,z,w]/(f)$, where $f = x^2y + y^{n-1} + zw$ and the grading of A is given by

$$(\deg x, \deg y, \deg z, \deg w) = (n-2, 2, n-1, n-1).$$

We consider the set of isomorphism classes of indecomposable graded non-free MCM A-modules (up to degree shifting).

When n is odd, we may take the modules $\{M_l : 1 \le l \le p\}$, $\{N_l : 1 \le l \le p\}$, $\{X_l : 1 \le l \le p+1\}$, $\{Y_l : 1 \le l \le p\}$ and $\{B_1, B_2\}$ as representatives, where $p = \frac{n-3}{2}$ and the modules $M_l, N_l, X_l, Y_l, B_1, B_2$ have presentation matrices

$$\mathsf{M}_l = \begin{pmatrix} z & 0 & -xy & -y^{n-1-l} \\ 0 & z & -y^{l+1} & xy \\ x & y^{n-2-l} & w & 0 \\ y^l & -x & 0 & w \end{pmatrix} \qquad \mathsf{N}_l = \begin{pmatrix} z & 0 & -x & -y^{n-2-l} \\ 0 & z & -y^l & x \\ xy & y^{n-1-l} & w & 0 \\ y^{l+1} & -xy & 0 & w \end{pmatrix}$$

$$\mathsf{X}_l = \begin{pmatrix} z & 0 & -x & -y^{n-1-l} \\ 0 & z & -y^l & xy \\ xy & y^{n-1-l} & w & 0 \\ y^l & -x & 0 & w \end{pmatrix} \qquad \mathsf{Y}_l = \begin{pmatrix} z & 0 & -xy & -y^{n-1-l} \\ 0 & z & -y^l & x \\ x & y^{n-1-l} & w & 0 \\ y^l & -xy & 0 & w \end{pmatrix}$$

$$\mathsf{B}_1 = \begin{pmatrix} z & -(x^2+y^{n-2}) \\ y & w \end{pmatrix} \qquad \mathsf{B}_2 = \begin{pmatrix} z & -y \\ x^2+y^{n-2} & w \end{pmatrix}$$

When n is even, we may take the modules $\{M_l : 1 \le l \le p-1\}$, $\{N_l : 1 \le l \le p-1\}$, $\{X_l : 1 \le l \le p\}$, $\{Y_l : 1 \le l \le p\}$ and $\{B_1, B_2, C_-, C_+, D_-, D_+\}$ as representatives, where $p = \frac{n-2}{2}$, the modules $M_l, N_l, X_l, Y_l, B_1, B_2$ have presentation matrices as above, and the modules C_-, C_+, D_-, D_+ have presentation matrices

$$\mathsf{C}_- = \begin{pmatrix} z & -(x+iy^p) \\ y(x-iy^p) & w \end{pmatrix} \qquad \mathsf{C}_+ = \begin{pmatrix} z & -(x-iy^p) \\ y(x+iy^p) & w \end{pmatrix}$$

$$\mathsf{D}_- = \begin{pmatrix} z & -y(x+iy^p) \\ x-iy^p & w \end{pmatrix} \qquad \mathsf{D}_+ = \begin{pmatrix} z & -y(x-iy^p) \\ x+iy^p & w \end{pmatrix}$$

For each of these representatives, we view the presentation matrix as the matrix of a graded A-linear map $d_0 : L_1 \to L_0$ with respect to some chosen homogeneous bases $\{e_i\}$ for L_0 and $\{f_i\}$ for L_1, with degrees given by

$$(\deg e_1, \deg e_2, \deg e_3, \deg e_4) = (0, n-2-2l, 1, n-1-2l)$$
$$(\deg f_1, \deg f_2, \deg f_3, \deg f_4) = (n-1, 2n-3-2l, n, 2n-2-2l)$$

for the module M_l,

$$(\deg e_1, \deg e_2, \deg e_3, \deg e_4) = (0, n-2-2l, -1, n-3-2l)$$
$$(\deg f_1, \deg f_2, \deg f_3, \deg f_4) = (n-1, 2n-3-2l, n-2, 2n-4-2l)$$

for the module N_l,

$$(\deg e_1, \deg e_2, \deg e_3, \deg e_4) = (0, n-2-2l, -1, n-1-2l)$$
$$(\deg f_1, \deg f_2, \deg f_3, \deg f_4) = (n-1, 2n-3-2l, n-2, 2n-2-2l)$$

for the module X_l,

$$(\deg e_1, \deg e_2, \deg e_3, \deg e_4) = (0, n-2l, 1, n-1-2l)$$
$$(\deg f_1, \deg f_2, \deg f_3, \deg f_4) = (n-1, 2n-1-2l, n, 2n-2-2l)$$

for the module Y_l,

$$(\deg e_1, \deg e_2) = (0, n-3)$$
$$(\deg f_1, \deg f_2) = (n-1, 2n-4)$$

for the module B_1,

$$(\deg e_1, \deg e_2) = (0, 3-n)$$
$$(\deg f_1, \deg f_2) = (n-1, 2)$$

for the module B_2,

$$(\deg e_1, \deg e_2) = (0, -1)$$
$$(\deg f_1, \deg f_2) = (n-1, n-2)$$

for the modules C_- and C_+, and

$$(\deg e_1, \deg e_2) = (0, 1)$$
$$(\deg f_1, \deg f_2) = (n-1, n)$$

for the modules D_- and D_+.

The module M_l for n odd. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree ω. Then $\deg p_{ij} = \deg e_j - \deg e_i + \omega$ and $\deg c_{ij} = \deg f_j - \deg f_i + \omega$. In case $\omega = 0$, P and C must be of the form

$$P = \begin{pmatrix} p_{11} & 0 & 0 & p_{14}y^{p-l+1} \\ 0 & p_{22} & p_{23}y^{l-p} & 0 \\ 0 & p_{32}y^{p-l} & p_{33} & 0 \\ 0 & 0 & 0 & p_{44} \end{pmatrix}$$
$$C = \begin{pmatrix} c_{11} & 0 & 0 & c_{14}y^{p-l+1} \\ 0 & c_{22} & c_{23}y^{l-p} & 0 \\ 0 & c_{32}y^{p-l} & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i, j \leq 4$, and $p_{23} = c_{23} = 0$ if $l \neq p$. In case $\omega = 1$, P and C must be of the form

$$P = \begin{pmatrix} 0 & p_{12}y^{p-l+1} & p_{13}y & p_{14}x \\ p_{21} & 0 & 0 & p_{24}y \\ p_{31} & 0 & 0 & p_{34}y^{p-l+1} \\ 0 & p_{42} & p_{43} & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & c_{12}y^{p-l+1} & c_{13}y & c_{14}x \\ c_{21} & 0 & 0 & c_{24}y \\ c_{31} & 0 & 0 & c_{34}y^{p-l+1} \\ 0 & c_{42} & c_{43} & 0 \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i, j \leq 4$, and $p_{21} = c_{21} = p_{43} = c_{43} = 0$ if $l \neq p$ and $p_{14} = c_{14} = 0$ if $l \neq 1$. We must solve the equation $(D_s(\mathsf{M}_l)) = \mathsf{M}_l C_s - P_s \mathsf{M}_l$ for $s = 1, 2, 3$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form (with $\omega = 0$ for $s = 1$ and $\omega = 1$ for $s = 2, 3$). This gives $P_1 = P_1^0 + a_1 \Phi_0$ with $a_1 \in k$, where $\Phi_0 = I_4$ and

$$P_1^0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Moreover, $P_s = P_s^0 + a_s \Phi_1$ with $a_s \in k$ for $s = 2, 3$, and $a_2 = a_3 = 0$ if $l \neq p$, where

$$\Phi_1 = \begin{pmatrix} 0 & y & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$P_2^0 = \begin{pmatrix} 0 & 0 & y & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, P_3^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The relation $2xyD_1 + zD_2 - wD_3 = 0$ gives the equation $2xyP_1 + zP_2 - wP_3 = 0$ in $\mathrm{End}_A(M_l)$, i.e.

$$\begin{pmatrix} 2(a_1-1)xy & a_2yz - a_3yw & yz & 0 \\ -a_2z + a_3w & 2(a_1-1)xy & 0 & -yz \\ w & 0 & 2a_1xy & -a_2yz + a_3yw \\ 0 & -w & a_2y - a_3w & 2a_1xy \end{pmatrix} = 0$$

By inspection, we see that this is a contradiction for $1 \leq l \leq p$.

The module N_l for n odd. We see that $N_l \cong M_l^\vee$ for $1 \leq l \leq p$ using lemma 11. It follows from lemma 4 and the computation above for M_l that the module N_l cannot admit a g-connection for $1 \leq l \leq p$.

The module X_l for n odd. We see that $X_l \cong Y_l^\vee$ for $1 \leq l \leq p$ using lemma 11. It follows from lemma 4 and the computation below for Y_l that the module X_l cannot admit a g-connection for $1 \leq l \leq p$. Since we include the case $l = p+1$ in the calculations below for Y_l, it also follows that the module $X_{p+1} \cong Y_{p+1}$ cannot admit a g-connection.

The module Y_l for n odd. In this case, we include the module Y_l for $1 \leq l \leq p+1$ for reasons mentioned above. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree ω. Then $\deg p_{ij} = \deg e_j - \deg e_i + \omega$ and $\deg c_{ij} = \deg f_j - \deg f_i + \omega$. In case $\omega = 0$, P and C must be of the form

$$P = \begin{pmatrix} p_{11} & p_{12}x & 0 & p_{14}y^{p-l+1} \\ 0 & p_{22} & p_{23} & 0 \\ 0 & p_{32}y^{p+1-l} & p_{33} & 0 \\ p_{41} & 0 & 0 & p_{44} \end{pmatrix}$$
$$C = \begin{pmatrix} c_{11} & c_{12}x & 0 & c_{14}y^{p-l+1} \\ 0 & c_{22} & c_{23} & 0 \\ 0 & c_{32}y^{p+1-l} & c_{33} & 0 \\ c_{41} & 0 & 0 & c_{44} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i, j \leq 4$, and $p_{23} = c_{23} = p_{41} = c_{41} = 0$ if $l \neq p+1$, $p_{12} = c_{12} = 0$ if $l \neq 1$. In case $\omega = 1$, P and C must be of the form

$$P = \begin{pmatrix} 0 & p_{12}y^{p-l+2} + p'_{12}z + p''_{12}w & p_{13}y & p_{14}x \\ p_{21} & 0 & 0 & p_{24} \\ p_{31} & p_{32}x & 0 & p_{34}y^{p+1-l} \\ 0 & p_{42}y & p_{43}y^{l-p} & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & c_{12}y^{p-l+2} + c'_{12}z + c''_{12}w & c_{13}y & c_{14}x \\ c_{21} & 0 & 0 & c_{24} \\ c_{31} & c_{32}x & 0 & c_{34}y^{p+1-l} \\ 0 & c_{42}y & c_{43}y^{l-p} & 0 \end{pmatrix}$$

with $p_{ij}, p'_{ij}, p''_{ij}, c_{ij}, c'_{ij}, c''_{ij} \in k$ for $1 \leq i, j \leq 4$, and $p_{43} = c_{43} = 0$ if $l < p$, $p'_{12} = p''_{12} = c'_{12} = c''_{12} = p_{14} = c_{14} = p_{32} = c_{32} = 0$ if $l \neq 1$, $p_{21} = c_{21} = 0$ if $l \neq p+1$. For $s = 1, 2, 3$, we must solve the equation $(D_s(\mathsf{Y}_l)) = \mathsf{Y}_l C_s - P_s \mathsf{Y}_l$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form (with $\omega = 0$ for $s = 1$ and $\omega = 1$ for $s = 2, 3$). This gives $P_1 = P_1^0 + a_1 \Phi_0$ with $a_1 \in k$, where $\Phi_0 = I_4$ and

$$P_1^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Moreover, $P_s = P_s^0 + a_s \Psi_1 + a'_s \Psi'_1 + a''_s \Psi''_1$ for $s = 2, 3$, and $a_s = a'_s = 0$ if $l \neq 1$, $a''_s = 0$ if $l \neq p+1$, where

$$\Psi_1 = \begin{pmatrix} 0 & -w & -y & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Psi'_1 = \begin{pmatrix} 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix}, \Psi''_1 = \begin{pmatrix} 0 & y & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & y & 0 \end{pmatrix}$$

and

$$P_2^0 = \begin{pmatrix} 0 & 0 & y & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, P_3^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix}$$

The relation $2xyD_1 + zD_2 - wD_3 = 0$ give the equation $2xyP_1 + zP_2 - wP_3 = 0$ in $\mathrm{End}_A(Y_l)$, i.e.

$$\begin{pmatrix} 2a_1xy & -bw + b'z + b''y & yz - by & bx \\ -b'' & 2a_1xy & 0 & -1 \\ w & b'x & 2(a_1+1)xy & -b'' \\ 0 & -yw + b'y & ab''y & 2(a_1+1)xy \end{pmatrix} = 0,$$

where $b = a_2z - a_3w$, $b' = a'_2z - a'_3w$ and $b'' = a''_2z - a''_3w$. By inspection, we see that this is a contradiction for $1 \leq l \leq p+1$.

The module B_1 for n odd. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree ω. Then $\deg p_{ij} = \deg e_j - \deg e_i + \omega$ and $\deg c_{ij} = \deg f_j - \deg f_i + \omega$. In case $\omega = 0$, P and C must be of the form

$$P = \begin{pmatrix} p_{11} & p_{12}y^p \\ 0 & p_{22} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12}y^p \\ 0 & c_{22} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i,j \leq 2$. In case $\omega = n-3$, P and C must be of the form

$$P = \begin{pmatrix} p_{11}y^p & p_{12}y^{n-3} + p'_{12}y^{p-1}w + p''_{12}y^{p-1}z \\ p_{21} & p_{22}y^p \end{pmatrix}$$
$$C = \begin{pmatrix} c_{11}y^p & c_{12}y^{n-3} + c'_{12}y^{p-1}w + c''_{12}y^{p-1}z \\ c_{21} & c_{22}y^p \end{pmatrix}$$

with $p_{ij}, p'_{ij}, p''_{ij}, c_{ij}, c'_{ij}, c''_{ij} \in k$ for $1 \leq i,j \leq 2$. We must solve the equation $(D_s(\mathsf{B}_1)) = \mathsf{B}_1 C_s - P_s \mathsf{B}_1$ for $s = 1,4,5$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form (with $\omega = 0$ for $s = 1$ and $\omega = n-3$ for $s = 4,5$). This gives $P_1 = P_1^0 + a_1\Phi_0$ with $a_1 \in k$, where $\Phi_0 = I_2$ and

$$P_1^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Moreover, $P_s = P_s^0 + a_s\Phi_{n-3}$ with $a_s \in k$ for $s = 4,5$, where we have $\Phi_{n-3} = y^p I_2$, $\gamma = (n-2)y^{n-3}$, and

$$P_4^0 = \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix}, \; P_5^0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let $\beta = x^2 + (n-1)y^{n-2}$. The relation $\beta D_1 + zD_4 - wD_5 = 0$ gives the equation $\beta P_1 + zP_4 - wP_5 = 0$ in $\operatorname{End}_A(B_1)$, i.e.

$$\begin{pmatrix} \beta + \delta & -z\gamma \\ -w & \delta \end{pmatrix} = 0,$$

where $\delta = a_1\beta + a_4 y^p z - a_5 y^p w$. By inspection, we see that this is a contradiction.

The module B_2 for n odd. We see that $B_2 \cong B_1^\vee$ using lemma 11. It follows from lemma 4 and the computation above for B_1 that the module B_2 cannot admit a g-connection.

The module M_l for n even. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree ω. Then $\deg p_{ij} = \deg e_j - \deg e_i + \omega$ and $\deg c_{ij} = \deg f_j - \deg f_i + \omega$. In case $\omega = 0$, P and C must be of the form

$$P = \begin{pmatrix} p_{11} & p_{12}y^{p-l} & 0 & 0 \\ 0 & p_{22} & 0 & 0 \\ 0 & 0 & p_{33} & p_{34}y^{p-l} \\ 0 & 0 & 0 & p_{44} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12}y^{p-l} & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & c_{34}y^{p-l} \\ 0 & 0 & 0 & c_{44} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i,j \leq 4$. In case $\omega = 1$, P and C must be of the form

$$P = \begin{pmatrix} 0 & 0 & p_{13}y & p_{14}y^{p+1-l} + p'_{14}x \\ 0 & 0 & p_{23} & p_{24}y \\ p_{31} & p_{32}y^{p-l} & 0 & 0 \\ 0 & p_{42} & 0 & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & 0 & c_{13}y & c_{14}y^{p+1-l} + c'_{14}x \\ 0 & 0 & c_{23} & c_{24}y \\ c_{31} & c_{32}y^{p-l} & 0 & 0 \\ 0 & c_{42} & 0 & 0 \end{pmatrix}$$

with $p_{ij}, p'_{ij}, c_{ij}, c'_{ij} \in k$ for $1 \leq i,j \leq 4$, and $p'_{14} = c'_{14} = 0$ if $l \neq 1$, $p_{23} = c_{23} = 0$ if $l \neq p-1$. For $s = 1,2,3$, we must solve the equation $(D_s(\mathsf{M}_l)) = \mathsf{M}_l C_s - P_s \mathsf{M}_l$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and

C_s have the above form (with $\omega = 0$ for $s = 1$ and $\omega = 1$ for $s = 2, 3$). This gives $P_1 = P_1^0 + a_1\Phi_0$ with $a_1 \in k$, where $\Phi_0 = I_4$ and

$$P_1^0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Moreover, $P_s = P_s^0$ for $s = 2, 3$, where

$$P_2^0 = \begin{pmatrix} 0 & 0 & y & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, P_3^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The relation $2xyD_1 + zD_2 - wD_3 = 0$ gives the equation $2xyP_1 + zP_2 - wP_3 = 0$ in $\mathrm{End}_A(M_l)$, i.e.

$$\begin{pmatrix} 2(a_1-1)xy & 0 & yz & 0 \\ 0 & 2(a_1-1)xy & 0 & -yz \\ w & 0 & 2a_1xy & 0 \\ 0 & -w & 0 & 2a_1xy \end{pmatrix} = 0$$

By inspection, we see that this is a contradiction for $1 \le l \le p-1$.

The module N_l for n even. We see that $N_l \cong M_l^\vee$ for $1 \le l \le p-1$ using lemma 11. It follows from lemma 4 and the computation above for M_l that the module N_l cannot admit a g-connection.

The module X_l for n even. We see that $X_l \cong Y_l^\vee$ for $1 \le l \le p$ using lemma 11. It follows from lemma 4 and the computation below for Y_l that the module X_l cannot admit a g-connection.

The module Y_l for n even. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree ω. Then $\deg p_{ij} = \deg e_j - \deg e_i + \omega$ and $\deg c_{ij} = \deg f_j - \deg f_i + \omega$. In case $\omega = 0$, P and C must be of the form

$$P = \begin{pmatrix} p_{11} & p_{12}y^{p+1-l} + p'_{12}x & 0 & 0 \\ 0 & p_{22} & 0 & 0 \\ 0 & 0 & p_{33} & p_{34}y^{p-l} \\ 0 & 0 & p_{43} & p_{44} \end{pmatrix}$$
$$C = \begin{pmatrix} c_{11} & c_{12}y^{p+1-l} + c'_{12}x & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & c_{34}y^{p-l} \\ 0 & 0 & c_{43} & c_{44} \end{pmatrix}$$

with $p_{ij}, p'_{ij}, c_{ij}, c'_{ij} \in k$ for $1 \leq i,j \leq 4$, and $p'_{12} = c'_{12}$ if $l \neq 1$, $p_{43} = c_{43} = 0$ if $l \neq p$. In case $\omega = 1$, P and C must be of the form

$$P = \begin{pmatrix} 0 & p_{12}z + p'_{12}w & p_{13}y + p'_{13}x & p_{14}y^{p+1-l} + p'_{14}x \\ 0 & 0 & p_{23} & p_{24} \\ p_{31} & p_{32}y^{p+1-l} + p'_{32}x & 0 & 0 \\ p_{41} & p_{42}y + p'_{42}x & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & c_{12}z + c'_{12}w & c_{13}y + c'_{13}x & c_{14}y^{p+1-l} + c'_{14}x \\ 0 & 0 & c_{23} & c_{24} \\ c_{31} & c_{32}y^{p+1-l} + c'_{32}x & 0 & 0 \\ c_{41} & c_{42}y + c'_{42}x & 0 & 0 \end{pmatrix}$$

with $p_{ij}, p'_{ij}, c_{ij}, c'_{ij} \in k$ for $1 \leq i,j \leq 4$, and $p_{23} = c_{23} = p_{41} = c_{41} = 0$ if $l \neq p$, $p_{12} = p'_{12} = c_{12} = c'_{12} = p'_{14} = c'_{14} = p'_{32} = c'_{32} = 0$ if $l \neq 1$, and furthermore $p'_{13} = c'_{13} = p'_{42} = c'_{42} = 0$ if $n \neq 4$. For $s = 1,2,3$, we must solve the equation $(D_s(\mathsf{Y}_l)) = \mathsf{Y}_l C_s - P_s \mathsf{Y}_l$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form (with $\omega = 0$ for $s = 1$ and $\omega = 1$ for $s = 2,3$). This gives $P_1 = P_1^0 + a_1\Phi_0$ with $a_1 \in k$, where $\Phi_0 = I_4$ and

$$P_1^0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Moreover, $P_s = P_s^0 + a_s\Psi_1 + a'_s\Psi'_1$ with $a_s, a'_s \in k$ for $s = 2,3$, and $a_s = a'_s = 0$ if $l \neq 1$, where

$$\Psi_1 = \begin{pmatrix} 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix}, \Psi'_1 = \begin{pmatrix} 0 & -w & -y & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$P_2^0 = \begin{pmatrix} 0 & 0 & y & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, P_3^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix}$$

The relation $2xyD_1 + zD_2 - wD_3 = 0$ gives the equation $2xyP_1 + zP_2 - wP_3 = 0$ in $\mathrm{End}_A(Y_l)$, i.e.

$$\begin{pmatrix} 2(a_1-1)xy & bz - b'w & yz - b'y & b'x \\ 0 & 2(a_1-1)xy & 0 & -z \\ w & bx & 2a_1xy & 0 \\ 0 & -yw + by & 0 & 2a_1xy \end{pmatrix} = 0,$$

where $b = a_2z - a_3w$ and $b' = a'_2z - a'_3w$. By inspection, we see that this is a contradiction for $1 \leq l \leq p$.

The module B_1 for n even. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree ω. Then $\deg p_{ij} = \deg e_j - \deg e_i + \omega$ and $\deg c_{ij} = \deg f_j - \deg f_i + \omega$. In case $\omega = 0$, P and C must be of the form

$$P = \begin{pmatrix} p_{11} & 0 \\ 0 & p_{22} \end{pmatrix}, C = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i, j \leq 2$. In case $\omega = n-3$, P and C must be of the form

$$P = \begin{pmatrix} 0 & p_{12}y^{n-3} + p'_{12}xy^{p-1} \\ p_{21} & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & c_{12}y^{n-3} + c'_{12}xy^{p-1} \\ c_{21} & 0 \end{pmatrix}$$

with $p_{ij}, p'_{ij}, c_{ij}, c'_{ij} \in k$ for $1 \leq i, j \leq 2$. For $s = 1, 4, 5$, we must solve the equation $(D_s(\mathsf{B}_1)) = \mathsf{B}_1 C_s - P_s \mathsf{B}_1$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form (with $\omega = 0$ for $s = 1$ and $\omega = n-3$ for $s = 4, 5$). This gives $P_1 = P_1^0 + a_1 \Phi_0$ with $a_1 \in k$, where $\Phi_0 = I_2$ and

$$P_1^0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Moreover, $P_s = P_s^0$ for $s = 4, 5$, where $\gamma = (n-2)y^{n-3}$ and

$$P_4^0 = \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix}, \; P_5^0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let $\beta = x^2 + (n-1)y^{n-2}$. The relation $\beta D_1 + zD_4 - wD_5 = 0$ gives the equation $\beta P_1 + zP_4 - wP_5 = 0$ in $\mathrm{End}_A(B_1)$, i.e.

$$\begin{pmatrix} -\beta + \delta & -z\gamma \\ -w & \delta \end{pmatrix} = 0,$$

where $\delta = a_1 \beta$. By inspection, we see that this is a contradiction.

The module B_2 for n even. We see that $B_2 \cong B_1^\vee$ using lemma 11. It follows from lemma 4 and the computation above for B_1 that the module B_2 cannot admit a g-connection.

The module C_- for n even. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree ω. Then $\deg p_{ij} = \deg e_j - \deg e_i + \omega$ and $\deg c_{ij} = \deg f_j - \deg f_i + \omega$. In case $\omega = 0$, P and C must be of the form

$$P = \begin{pmatrix} p_{11} & 0 \\ 0 & p_{22} \end{pmatrix}, \; C = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i, j \leq 2$. In case $\omega = n-3$, P and C must be of the form

$$P = \begin{pmatrix} 0 & p_{12}y^{p-1} \\ p_{21}x + p'_{21}y^p & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & c_{12}y^{p-1} \\ c_{21}x + c'_{21}y^p & 0 \end{pmatrix}$$

with $p_{ij}, p'_{ij}, c_{ij}, c'_{ij} \in k$ for $1 \leq i, j \leq 2$. For $s = 1, 4, 5$, we must solve the equation $(D_s(\mathsf{C}_-)) = \mathsf{C}_- C_s - P_s \mathsf{C}_-$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form (with $\omega = 0$ for $s = 1$ and $\omega = n-3$ for $s = 4, 5$). This gives $P_1 = P_1^0 + a_1 \Phi_0$ with $a_1 \in k$, where $\Phi_0 = I_2$ and

$$P_1^0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Moreover, $P_s = P_s^0$ for $s = 4, 5$, where

$$P_4^0 = \begin{pmatrix} 0 & ipy^{p-1} \\ 0 & 0 \end{pmatrix}, \; P_5^0 = \begin{pmatrix} 0 & 0 \\ -x + i(p+1)y^p & 0 \end{pmatrix}$$

Let $\beta = x^2 + (n-1)y^{n-2}$. The relation $\beta D_1 + zD_4 - wD_5 = 0$ gives the equation $\beta P_1 + zP_4 - wP_5 = 0$ in $\mathrm{End}_A(C_-)$, i.e.

$$\begin{pmatrix} (a_1-1)\beta & ipy^{p-1}z \\ xw - i(p+1)y^p w & a_1\beta \end{pmatrix} = 0$$

By inspection, we see that this is a contradiction.

The module C_+ for n even. Let P and C be matrices corresponding to graded endomorphisms of L_0 and L_1 of degree ω. Then $\deg p_{ij} = \deg e_j - \deg e_i + \omega$ and $\deg c_{ij} = \deg f_j - \deg f_i + \omega$. In case $\omega = 0$, P and C must be of the form

$$P = \begin{pmatrix} p_{11} & 0 \\ 0 & p_{22} \end{pmatrix}, \; C = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix}$$

with $p_{ij}, c_{ij} \in k$ for $1 \leq i, j \leq 2$. In case $\omega = n-3$, P and C must be of the form

$$P = \begin{pmatrix} 0 & p_{12}y^{p-1} \\ p_{21}x + p'_{21}y^p & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & c_{12}y^{p-1} \\ c_{21}x + c'_{21}y^p & 0 \end{pmatrix}$$

with $p_{ij}, p'_{ij}, c_{ij}, c'_{ij} \in k$ for $1 \leq i, j \leq 2$. For $s = 1, 4, 5$, we must solve the equation $(D_s(\mathsf{C}_+)) = \mathsf{C}_+ C_s - P_s \mathsf{C}_+$, where D_s is the derivation of A mentioned in the proof of theorem 16, and P_s and C_s have the above form (with $\omega = 0$ for $s = 1$ and $\omega = n-3$ for $s = 4, 5$). This gives $P_1 = P_1^0 + a_1\Phi_0$ with $a_1 \in k$, where $\Phi_0 = I_2$ and

$$P_1^0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Moreover, $P_s = P_s^0$ for $s = 4, 5$, where

$$P_4^0 = \begin{pmatrix} 0 & -ipy^{p-1} \\ 0 & 0 \end{pmatrix}, \; P_5^0 = \begin{pmatrix} 0 & 0 \\ -x - i(p+1)y^p & 0 \end{pmatrix}$$

Let $\beta = x^2 + (n-1)y^{n-2}$. The relation $\beta D_1 + zD_4 - wD_5 = 0$ gives the equation $\beta P_1 + zP_4 - wP_5 = 0$ in $\mathrm{End}_A(C_+)$, i.e.

$$\begin{pmatrix} (a_1-1)\beta & -ipy^{p-1}z \\ xw + i(p+1)y^p w & a_1\beta \end{pmatrix} = 0$$

By inspection, we see that this is a contradiction.

The modules D_- and D_+ for n even. We see that $D_- \cong C_+^\vee$ and that $D_+ \cong C_-^\vee$ using lemma 11. It follows from lemma 4 and the computations above for C_- and C_+ that the modules D_- and D_+ cannot admit g-connections.

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